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Models of Two-Level Atoms in Quasiperiodic External Fields

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We prove that the spectrum of the generalized quasienergy operator of a plane-polarized two-level atom in a strong external quasiperiodic electromagnetic field with nonzero constant Fourier component is pure point, under Diophantine conditions on the frequency ratio, and excluding a small subset of resonant values. The widespread belief that there may be only pure point spectrum in such models is briefly discussed in Section 2 and the circularly polarized case—a well-known soluble model—is revisited from the point of view of the quasienergy operator..

KEY WORDS: Two-level atoms; quasiperiodic external fields; quasienergy operator; spectrum; quantum stability.

1. THE PLANE POLARIZED CASE

Quantum stability has been the subject of intensive research since the basic papers of Bellissard[1] and Cornbescure.[2] More recently, interest arose in quantum systems under quasiperiodic perturbations[3–7] in the context of "quantum chaos," because quantum localization effects seem to be less pronounced in such systems, in comparison with their periodic counterparts: see the review,[8] which is a stimulating introduction to the subject.

We consider models of two-level atoms, interacting with a sinusoidal external bichromatic electromagnetic field in the dipole approximation.[9] For plane-polarized external fields, the Hamiltonian is

$$H = \beta\sigma_z + [\lambda_1 \cos(\omega_1 t) + \lambda_2 \cos(\omega_2 t)]\,\sigma_x \tag{1}$$

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Above, $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices, (2β) denotes the energy difference between the (two) atomic levels, ω_1 and ω_2 are the field frequencies (in general, incommensurate), and λ_1, λ_2 are real coupling constants (without loss positive). The function $f(\omega_1 t, \omega_2 t) = \lambda_1 \cos(\omega_1 t) + \lambda_2 \cos(\omega_2 t)$ is the simplest example of a quasiperiodic function.[10] It is thus convenient to generalize (1) to

$$H = \beta\sigma_z + \lambda f(\omega_1 t, \omega_2 t)\,\sigma_x \tag{2}$$

We assume that f is a quasi periodic function, with $f(\theta_1, \theta_2)$ *analytic* in a strip

$$\{(\theta_1, \theta_2)/\mathrm{Im}\,\theta_i < r_0,\ i = 1, 2\} \tag{3}$$

The corresponding generalized quasienergy operator[3] is

$$K = -i\left(\omega_1 \frac{\partial}{\partial\theta_1} + \omega_2 \frac{\partial}{\partial\theta_2}\right) + \beta\sigma_z + \lambda f(\theta_1, \theta_2)\,\sigma_x$$

on the (extended) Hilbert space

$$\mathscr{H} = \mathbb{C}^2 \otimes L^2(\mathbb{T}^2, d\mu)$$

where $\mathbb{T}^2$ denotes the (two-dimensional) torus and $d\mu = 1/(2\pi)^2\, d\theta_1, d\theta_2$ is the corresponding ergodic measure.

This case covers all the physical applications so far.

For small coupling (λ small in (2)) it was proved in ref. 4 that the spectrum of the quasienergy operator corresponding to (2) is pure point, when λ is excluded from a "small" set of resonant values (see ref. 4 for the precise statement) The method of proof was an operator KAM perturbation technique developed by Bellissard[1] and Combescure.[2]

The other domain of coupling constants which seems to be amenable to KAM-type perturbation is that of *large* coupling. Our presentation of the results in this section differs from that in ref. 4, and follows in part the beautiful paper of Howland.[11]

The quasienergy operator corresponding to (2) in the case of large coupling (i.e. $\beta = \varepsilon$ "small" and $\lambda = \text{fixed} = 1$) is the operator on $\mathscr{H}$ defined by

$$K_1 = -i\left(\omega_1 \frac{\partial}{\partial\theta_1} + \omega_2 \frac{\partial}{\partial\theta_2}\right) + \varepsilon\sigma_z + f(\theta_1, \theta_2)\,\sigma_x \tag{4}$$

It is convenient to consider the limiting case ("infinite coupling")

$$\tilde{K}_1 = -i\left(\omega_1 \frac{\partial}{\partial\theta_1} + \omega_2 \frac{\partial}{\partial\theta_2}\right) + f(\theta_1, \theta_2)\, \sigma_x \tag{5}$$

In this case the Schrödinger equation is

$$i\frac{\partial\psi}{\partial t} = f(\omega_1 t, \omega_2 t)\, \sigma_x \psi \tag{6a}$$

with solution

$$\psi(t) = \exp\left[-i\int_0^t f(\omega_1\tau, \omega_2\tau)\, d\tau\, \sigma_x\right]\psi(0) \tag{6b}$$

We have the following theorem of Bohr:[12]

Proposition 1. ψ, given by (6), is quasiperiodic *iff*

$$\int_0^t f(\omega_1\tau, \omega_2\tau)\, d\tau = ct + \varphi(t) \tag{7}$$

where φ is quasiperiodic

Writing

$$f(\theta_1, \theta_2) = \sum_{n_1, n_2 \in Z} f_{n_1 n_2} e^{i(n_1\theta_1 + n_2\theta_2)} \tag{8}$$

then

$$c = f_{0,0} \tag{9}$$

Alternatively, from the point of view of the generalized quasi-energy operator $\tilde{K}_1$, we have, after a rotation,

$$K_1' = -i\left(\omega_1 \frac{\partial}{\partial\theta_1} + \omega_2 \frac{\partial}{\partial\theta_2}\right) + f(\theta_1, \theta_2)\, \sigma_z = K_+ \oplus K_-$$

where

$$K_\pm = -i\left(\omega_1 \frac{\partial}{\partial\theta_1} + \omega_2 \frac{\partial}{\partial\theta_2}\right) \pm f(\theta_1, \theta_2) \tag{10}$$

on $C \otimes L^2(\mathbb{T}^2)$. It was proved in ref. 4 that if $|\alpha n_1 + n_2| > \gamma/|n_1|^\sigma$ for $\gamma > 0$, $\sigma > 1$ $\forall \underset{\sim}{n} \in Z^2 (n_1 \neq 0)$, with $\alpha = \omega_1/\omega_2$ (a diophantine condition), and

$$\sum_{\mathbf{n} \in \mathbb{Z}^2} |f_\mathbf{n}| \, |n_1|^\sigma < \infty \tag{11}$$

the generalized Floquet operator (refs. 3 and 4) has pure point spectrum. (The single condition $\sum_{\mathbf{n} \in \mathbb{Z}^2} |f_\mathbf{n}/\mathbf{n} \cdot \boldsymbol{\omega}| < \infty$ in ref. 4 is somewhat misleading because it implies $f_\mathbf{0} = 0$, which is not necessary, because it is not the integral of f which must be quasiperiodic, but the exponential of the integral, see Proposition 1).

From the point of view of the generalized quasi energy operator this example was discussed by Howland[11] in a slightly different context:

Proposition 2.[11]. Assume that

$$\sum_{n_1, n_2 \in \mathbb{Z}} |f_{n_1 n_2}| \, |\log |f_{n_1 n_2}|| < \infty \tag{12}$$

Then $K_\pm$ in (10) are unitarily equivalent to

$$\tilde{K}_\pm = -i \left(\omega_1 \frac{\partial}{\partial \theta_1} + \omega_2 \frac{\partial}{\partial \theta_2} \right) \pm f_{00} \tag{13}$$

for almost every $\alpha = \omega_2/\omega_1$. Notice that (12) is weaker than (11).

The proof consists in introducing the function

$$v(\theta_1, \theta_2) = \sum_{\substack{n_1 n_2 \\ (n_1, n_2) \neq (0, 0)}} \frac{f_{n_1 n_2}}{n_1 + \alpha n_2} e^{i(n_1 \theta_1 + n_2 \theta_2)} \tag{14}$$

and showing that the above series is absolutely convergent, with $v \in C^1(\mathbb{T}^2)$, and

$$-i \left(\omega_1 \frac{\partial}{\partial \theta_1} + \omega_2 \frac{\partial}{\partial \theta_2} \right) v(\theta_1, \theta_2) = f(\theta_1, \theta_2) - f_{00}$$

Hence

$$e^{iv} K_\pm e^{-iv} = \tilde{K}_\pm$$

A simple generalization of this argument (performing a unitary gauge transformation) is applicable to (4) (see also ref. 13, where this generalization was applied to a modification of (1)). Rotating K given by (4), through $\pi/2$ about the y axis, we obtain

$$K'_1 = -i\left(\omega_1 \frac{\partial}{\partial\theta_1} + \omega_2 \frac{\partial}{\partial\theta_2}\right) + \varepsilon\sigma_x + f(\theta_1, \theta_2)\,\sigma_z \tag{15}$$

Introducing

$$\tilde{v}_{n_1 n_2} = -\frac{f_{n_1 n_2}}{i(\omega_1 n_1 + \omega_2 n_2)} \tag{16}$$

and

$$U = \begin{pmatrix} e^{i\tilde{v}} & 0 \\ 0 & e^{-i\tilde{v}} \end{pmatrix} \tag{17}$$

we see that

$$U^{-1} K'_1 U = K''_1 \tag{18}$$

where

$$K''_1 = -i\omega_1 \frac{\partial}{\partial\theta_1} - i\omega_2 \frac{\partial}{\partial\theta_2} + \varepsilon \begin{pmatrix} 0 & e^{-2i\tilde{v}} \\ e^{2i\tilde{v}} & 0 \end{pmatrix} + f_{00}\sigma_z \tag{19}$$

According to propositions 1 and 2, and (9), there are two cases:

(a) $f_{00} \neq 0$

(b) $f_{00} = 0$

The following theorem is a simple generalization of the argument of ref. 13:

Theorem 1. In case (a), if f is analytic in a strip (3), the spectrum of K_1, given by (4), is pure point, if α is excluded from a "small" subset of resonant values. (Precise conditions are given in theorem 6.1 of ref. 4 and are the same here)

Proof. We apply theorem 6.1 of ref. 4 to K''_1 given by (19). The only difference, as in ref. 13, is that the "potential"

$$V(\theta_1, \theta_2) = \begin{pmatrix} 0 & e^{-2i\tilde{v}} \\ e^{2i\tilde{v}} & 0 \end{pmatrix}$$

depends on α by (16), and it must be verified that $\|V\|_{r,\Omega} < \infty$, where $\|\cdot\|_{r,\Omega}$ denotes the Combescure norm (refs. 2 and 4), $r < r_0$, and $\Omega = (1, \infty)$.

This is an exercise in complex analysis, using the assumption of analyticity of f and definition (16).

2. NONPERTURBATIVE RESULTS ON THE SPECTRUM: CONCLUSION AND OPEN PROBLEMS

There is a widespread belief that there is neither a spectral transition, or mixed spectrum in models of two-level atoms with analytic perturbations, such as the one described in Section 1, i.e., the spectrum is expected to be pure point, under diophantine conditions on the frequency ratio.

In this connection, we remark that the *circularly polarized* case corresponding to (1):

$$H = \beta\sigma_z + \lambda_1[\cos(\omega_1 t)\,\sigma_x + \sin(\omega_1 t)\,\sigma_y] + \lambda_2[\cos(\omega_2 t)\,\sigma_x + \sin(\omega_2 t)\,\sigma_y] \tag{20}$$

is such that the corresponding generalized quasienergy operator

$$K = -i\left(\omega_1 \frac{\partial}{\partial\theta_1} + \omega_2 \frac{\partial}{\partial\theta_2}\right) + \beta\sigma_z + \lambda_1(\cos\theta_1\sigma_x + \sin\theta_1\sigma_y) + \lambda_2(\cos\theta_2\sigma_x + \sin\theta_2\sigma_y) \tag{21}$$

is pure point for all λ_1, λ_2, i.e., a nonperturbative result. Model (20) is a well-known soluble model (see ref. 14 for reviews): each term of type $\{\lambda[\cos(\omega t)\,\sigma_x + \sin(\omega t)\,\sigma_y]\} = \lambda e^{-i\omega t}\sigma_+ + \lambda e^{i\omega t}\sigma_-$ where $\sigma_+ = \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)$ and $\sigma_- = \left(\begin{smallmatrix}0 & 0\\ 1 & 0\end{smallmatrix}\right)$ are the spin-flip operators, corresponds to performing the "rotating-wave approximation" (RWA), whereby antiresonant terms are neglected (ref. 9, p. 144). we have:

Proposition 3. The spectrum of K is pure point for all $\omega_1, \omega_2, \lambda_1$ and λ_2.

Proof. We write

$$\tilde{K} = \begin{pmatrix} e^{i\theta_1} & 0\\ 0 & 1\end{pmatrix} K \begin{pmatrix} e^{-i\theta_1} & 0\\ 0 & 1\end{pmatrix} \tag{22}$$

where

$$\tilde{K} = -i\left(\omega_1 \frac{\partial}{\partial\theta_1} + \omega_2 \frac{\partial}{\partial\theta_2}\right) + \omega_1 \begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix} + \beta\sigma_z + \lambda_1\sigma_x + \lambda_2[\cos(\theta_2 - \theta_1)\,\sigma_x + \sin(\theta_2 - \theta_1)\,\sigma_y] \tag{23}$$

Define, now, the maps $g(\theta, \bar{\theta}) \to (\theta_1, \theta_2)$ by $\{ {}^{\theta_1 = \bar{\theta} - \theta}_{\theta_2 = \theta} \cdot g$ maps the torus $\mathbb{T}^2$ into $\mathbb{T}^2$, and the induced map $(Wf)(\theta, \bar{\theta}) = f[g(\theta, \bar{\theta})]$ maps $L^2(\mathbb{T}^2, d\mu)$ into $L^2(\mathbb{T}^2, d\mu)$ isometrically, carrying $\tilde{K}$ into $\hat{K}$, given by

$$\hat{K} = -i\omega_2 \frac{\partial}{\partial \bar{\theta}} - i(\omega_2 - \omega_1) \frac{\partial}{\partial \theta} + \beta \sigma_z + \omega_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_1 \sigma_x + \lambda_2 (\cos \theta \sigma_x + \sin \theta \sigma_y) \tag{24}$$

More precisely, $\tilde{K}$ is equivalent under W to the restriction of $\hat{K}$ to the range of W. On $\mathscr{H} = \mathbb{C} \otimes L^2(\tilde{\mathbb{T}}^2, d\mu = (1/(2\pi)^2)\, d\theta\, d\bar{\theta})$, where $\tilde{\mathbb{T}}^2$ denotes the torus in the $(\theta, \bar{\theta})$ variables, $\hat{K}$, given by (24), is the sum of two *commuting* operators $\hat{K}_1$ and $\hat{K}_2$, where

$$\hat{K}_1 \equiv -i\omega_2 \frac{\partial}{\partial \bar{\theta}} \mathbf{1} \tag{25}$$

and $\hat{K}_2$ is the Floquet operator corresponding to the (periodic) Hamiltonian

$$H(t) = \beta \sigma_z + \omega_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_1 \sigma_x + \lambda_2 \{ \cos[(\omega_2 - \omega_1) t]\, \sigma_x + \sin[(\omega_2 - \omega_1) t]\, \sigma_y \}$$

Since $\hat{K}_1$ has point spectrum by ref. 15 (bounded energy), as well as $\hat{K}_2$ by (25), $\hat{K}$ has point spectrum, and hence the same holds for $\tilde{K}$.

The above result cannot be indicative of the general behaviour for analytic perturbations, because, in this case, the spectrum of the generalized quasienergy operator is pure point even for $\alpha = \omega_2/\omega_1$ rational, while such is not the case for plane polarization (for instance, in the special case (5), a simple proof exists[11] that $\tilde{K}_1$ is absolutely continuous, see also ref. 1]. Hence, more general nonperturbative results are required to arrive at a general picture. Existence of continuous spectrum for non analytic perturbations has been proved in refs. 4 and 7.

Theorem 1 shows that, in the case of plane polarization, the only unsolved case for large coupling is b). In this case, the unperturbed quasienergy operator has (doubly) degenerate spectrum. This problem is of general nature: KAM perturbation theory in the presence of (finite) degeneracies, and merits further study.

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